

# Coherent Transport of Levitons Through the Kondo Resonance

Takafumi J. Suzuki

*Department of Physics, The University of Tokyo,  
7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan*

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We study coherent transport of levitons through a single-level quantum dot system driven by Lorentzian-shaped voltage pulses. We demonstrate the repeated emergence of the Kondo resonance in the dynamical regimes where the Fermi sea is driven by optimal pulses free of particle-hole excitations. The formation of the Kondo resonance significantly enhances the dc transport of levitons.

*Introduction.*— The Kondo effect has been one of the central subjects of condensed matter physics over the past 50 years [1, 2]. It is an archetypal example of coherent many-body phenomena in interacting electronic systems, and the essential low-energy behaviors of Kondo systems are described by the local Fermi liquid theory. The development of nanotechnology has extended the Kondo physics to nonequilibrium regimes [3–6]. Transport through a quantum dot (QD) enables us to study the nonequilibrium Kondo effect with experimentally tuned parameters. The interplay between the coherent many-body resonance and the nonequilibrium field has posed nontrivial problems. In particular, determining the fundamental excitation of the Kondo system driven out of equilibrium remains a challenging topic [6–8]. Recently, new insights on the nonequilibrium Kondo physics have been gained in tandem with technological advances in engineering time-dependent fields. Electrons dressed with photons acquire novel features which equilibrium electrons do not possess. Periodic driving fields have been frequently utilized to probe the low energy excitations of the QD systems [9–19] and to invent new types of Kondo systems [20, 21]. In sinusoidally driven QD systems, satellites of the Kondo peak develop at low temperatures due to the absorption and the emission of photons [11, 12]. On the other hand, the spin-flip cotunneling processes [14] as well as the ionization of the local site [15] induce decoherence which hinders the Kondo resonance.

The richness and the complexity of the driven electronic systems come from the collective response of electrons under the entire Fermi sea. In spite of the difficulty originating from the many-body effects, Leviton and coauthors proposed an elegant way to engineer minimal excitation states out of the Fermi sea [22–24]. They found that, among possible pulse profiles, the repeated Lorentzian pulses excite the Fermi sea without creating particle-hole excitations. The single-particle nature of the excitation were clarified at the same time in terms of the full counting statistics. The elementary excitations created above the undisturbed Fermi sea are termed levitons, and have been experimentally exploited as ideal fermionic excitations in electron quantum optics [25, 26]. There have already been a number of works on levitons injected in quantum Hall systems [24, 27–31]. It is also

theoretically proposed that a Fermi sea driven by designed Lorentzian pulses hosts an exotic excitation with a fractional effective charge [32].

In this Letter, we demonstrate the coherent transport of levitons through the Kondo resonance in a QD system driven by Lorentzian-shaped periodic pulses. The Lorentzian driving protocol is distinct from the others because the optimal pulses can excite a fermionic quasiparticle while preserving the structure of the Fermi sea. This enables the coexistence of the Kondo resonance with the strong driving field. We use a many-body approach combined with Floquet’s formalism [33, 34] to provide a conceptually transparent and numerically efficient way to describe the dynamics of the interacting levitons.

*Setup.*— We consider a single-level QD coupled to left and right leads with a periodically oscillating bias voltage. The Hamiltonian reads

$$H = \sum_{\sigma} \epsilon_d \hat{d}_{\sigma}^{\dagger} \hat{d}_{\sigma} + \sum_{\alpha, \mathbf{k}, \sigma} (\epsilon_{\alpha \mathbf{k}} + eV_{\alpha}(t)) \hat{c}_{\alpha \mathbf{k} \sigma}^{\dagger} \hat{c}_{\alpha \mathbf{k} \sigma} + \sum_{\alpha, \mathbf{k}, \sigma} \left( t_{\alpha} \hat{d}_{\sigma}^{\dagger} \hat{c}_{\alpha \mathbf{k} \sigma} + \text{h.c.} \right) + U \hat{d}_{\uparrow}^{\dagger} \hat{d}_{\uparrow} \hat{d}_{\downarrow}^{\dagger} \hat{d}_{\downarrow}, \quad (1)$$

where  $\hat{d}_{\sigma}^{\dagger}$  creates an electron in the QD with spin  $\sigma$  and  $\hat{c}_{\alpha \mathbf{k} \sigma}^{\dagger}$  creates a conduction electron in the lead  $\alpha$  ( $= L, R$ ) with spin  $\sigma$  and momentum  $\mathbf{k}$ . The coupling  $t_{\alpha}$  between the QD and the lead  $\alpha$  causes the level broadening  $\Gamma_{\alpha} \equiv 2\pi |t_{\alpha}|^2 \rho_{\alpha}$ , where  $\rho_{\alpha}$  is the density of states (DOS) of the conduction electrons at the Fermi energy  $\epsilon_F$ . We consider that the left lead is irradiated with the repeated Lorentzian pulses

$$V_L(t) = \sum_{m=-\infty}^{\infty} \frac{V_{AC}}{\pi} \frac{T_p \tau_w}{(t - mT_p)^2 + \tau_w^2}, \quad (2)$$

with period  $T_p$ , width  $\tau_w$ , and amplitude  $V_{AC}$ . The right lead is in equilibrium, i.e.  $V_R(t) = 0$ .

The periodically driven QD system can be well described in the Floquet-Green’s function method [33, 34]. In the following, we drop the spin index  $\sigma$  for simplicity. The propagators of the photon-dressed interacting electrons are given by the retarded and the lesser Green’s functions  $G^r(t, t') = -i\theta(t - t')\langle d(t)d^{\dagger}(t') \rangle$  and  $G^<(t, t') \equiv i\langle d^{\dagger}(t')d(t) \rangle$ , respectively. Their Floquet representations are introduced as  $\mathbf{G}_{mn}^{r(<)}(\omega) \equiv$

$\int_{-\infty}^{\infty} dt \int_{-T_p/2}^{T_p/2} \frac{dT}{T_p} e^{i(\omega+m\hbar\Omega)t-i(\omega+n\hbar\Omega)t'} G^{r(<)}(t, t')$  with the driving frequency  $\Omega = 2\pi/T_p$ . Hereafter, we use bold letters to denote functions in the Floquet representation.

The equilibrium distribution in the right lead is written in the Floquet representation as  $\mathbf{f}_R = \mathbf{f}^{\text{eq}}$ , where  $\mathbf{f}_{mn}^{\text{eq}}(\omega) = \delta_{mn}/(e^{\beta(\omega+m\hbar\Omega-\epsilon_F)} + 1)$  with the Kronecker delta  $\delta_{mn}$ . In contrast, the time-dependent phase  $\varphi(t) = \frac{e}{\hbar} \int_{-\infty}^t V_L(t') dt'$  acquired by the electrons tunneling from the left lead to the QD significantly modifies the distribution function as  $\mathbf{f}_L(\omega) = \mathbf{U} \mathbf{f}^{\text{eq}}(\omega - eV_{\text{AC}}) \mathbf{U}^\dagger$ . Here, the dc offset of the periodic Lorentzian pulses is included as a shift of the chemical potential, and absorption and emission of photons are described via the unitary matrix  $U_{mn} = u_{m-n}$  with  $u_l \equiv \int_{-T_p/2}^{T_p/2} \frac{dt}{T_p} e^{i(l\hbar\Omega+eV_{\text{AC}})t} e^{-i\varphi(t)}$ .

For the repeated Lorentzian pulses (2), the matrix elements are computed as

$$u_l = \sum_{k=\max\{0, -l\}}^{\infty} \frac{\Gamma(k+l+q)\Gamma(k-q)e^{-2\pi\tau(2k+l)}}{\Gamma(q)\Gamma(k+l+1)\Gamma(-q)\Gamma(k+1)}, \quad (3)$$

with the Gamma function  $\Gamma(x)$ ,  $q \equiv eV_{\text{AC}}/\hbar\Omega$ , and  $\tau \equiv \tau_w/T_p$ . In the delta-pulse limit  $\tau \rightarrow 0$ ,  $\mathbf{f}_L$  is identical to the equilibrium distribution  $\mathbf{f}^{\text{eq}}$  at  $q \in \mathbb{Z}$  because each pulse introduces the  $2\pi q$  phase shift. This is a direct consequence of the gauge invariance. The crucial property of the quantized Lorentzian pulses is that they generate purely electronic excitations even for finite  $\tau$  with minimal disturbance of the Fermi sea [22–24]. Moreover, the probability to excite higher Fourier harmonics decay exponentially. They are in contrast to non-quantized pulses which inevitably excite a number of particle-hole pairs as is the case with the Anderson orthogonality catastrophe problem [22, 35]. These peculiar properties of the quantized Lorentzian pulses result in the quasiparticle nature of levitons created above the undisturbed Fermi sea [35, 36]. Since the propagators of the photon-dressed electrons have the same diagrammatic structure as those in equilibrium, properties of equilibrium interacting electrons are straightforwardly inherited by the levitons. In particular, the dot electron and the electrons under the undisturbed Fermi sea form the Kondo resonance through which the leviton flows. This is the central idea of this Letter.

*Dynamical formation of the Kondo Resonance.*— One of the hallmarks of the Kondo effect is the appearance of the resonant peak in the time-averaged DOS

$$\bar{\rho}(\omega) \equiv -\frac{1}{\pi} \text{Im} \int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} \frac{dT}{T_p} \int_{-\infty}^{\infty} dt_r e^{i\omega t_r} G^r(t, t'), \quad (4)$$

with  $t_r \equiv t - t'$  and  $T \equiv (t + t')/2$ . In the following, we evaluate the self-energy up to the second order in  $U$  [37] to illustrate qualitative behaviors of the interacting photon-dressed electrons. The retarded Green's function can be efficiently calculated in the

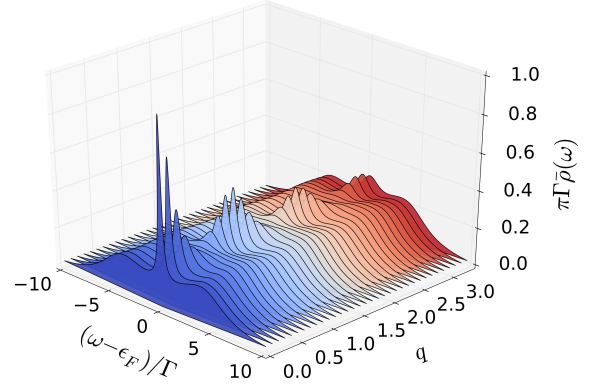


FIG. 1. (Color online) Time-averaged DOS with  $\Gamma_{L,R} = 1$ ,  $\epsilon_d = -4$ ,  $U = 8$ ,  $\beta = 100$ ,  $\tau = 0.02$ , and  $\hbar\Omega = 3$  for various values of  $q$ .

Floquet representation because it has a simple matrix form  $\mathbf{G}^r = [\mathbf{1} - \mathbf{g}^r \Sigma_U^r]^{-1} \mathbf{g}^r$  [33, 34]. Here, the exact propagators are constructed from the unperturbed one  $\mathbf{g}_{mn}^r(\omega) = \delta_{mn}/(\omega + m\hbar\Omega - E_d + i\Gamma)$  with the energy level  $E_d = \epsilon_d + U n_d$  and the linewidth  $\Gamma = (\Gamma_L + \Gamma_R)/2$ . The charge  $n_d$  is determined within the Hartree approximation as  $n_d = -\frac{1}{\pi} \text{Im} \sum_m \int d\omega \mathbf{g}_{mm}^<(\omega)$ , where the unperturbed lesser Green's function is given by  $\mathbf{g}^< = \mathbf{g}^r \Sigma_0^< \mathbf{g}^a$  with  $\Sigma_0^< = i(\Gamma_L \mathbf{f}_L + \Gamma_R \mathbf{f}_R)$ . The  $U^2$  term of the self-energy is given as  $\Sigma_U(z, z') = U^2 g(z, z') g(z', z) g(z, z')$  on the Keldysh contour [38]. The retarded and lesser components of the self-energy are obtained by projecting the Keldysh arguments  $z$  and  $z'$  onto the real-time axis.

Figure 1 shows the time-averaged DOS for various values of  $q$ . The impurity parameters are chosen as  $\Gamma_{L,R} = 1$ ,  $\epsilon_d = -4$ ,  $U = 8$ , and  $\beta = 100$ . The Lorentzian-shaped bias voltage with  $\tau = 0.02$  and  $\hbar\Omega = 3$  excites conduction electrons in the left lead, dressing them with a large numbers of photons. While the Kondo peak observed at  $q = 0$  is reduced by the irradiation, the resonant peak reappears at  $q = eV_{\text{AC}}/\hbar\Omega = 1$ , which is much larger than the Kondo temperature  $T_K/\hbar\Omega \sim 0.029$  estimated with the expression  $T_K = \sqrt{U\Gamma}/2 \exp[\pi\epsilon_d(\epsilon_d + U)/2U\Gamma]$  [39]. The reduction and the formation of the Kondo peak are periodically repeated around the larger integer values of  $q$ . The results are distinct from those of a sinusoidally driven QD, where the irradiation suppresses the Kondo resonance in the corresponding regimes [15, 16]. The appearance of the Kondo peak around  $q \in \mathbb{Z}$  originates from the aforementioned recovery of the Fermi sea in the dynamical regime: electrons in the minimally disturbed Fermi sea form the many-body resonance state with the dot electron. The reduction of the peak height at large integer  $q$  can be attributed to the imperfect formation of the Fermi sea.

The dynamical formation of the Kondo resonance can

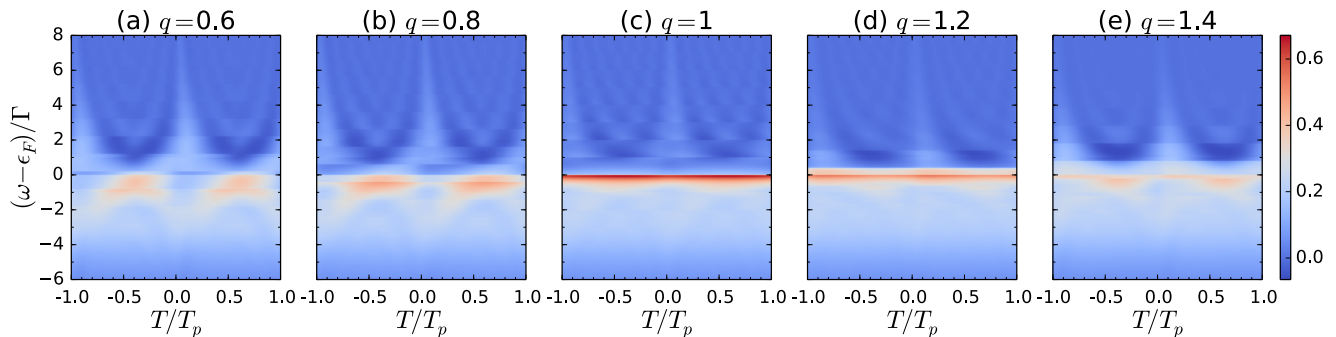


FIG. 2. (Color online) Density plots of the Wigner function for various values of  $q$ . The QD with  $\Gamma_{L,R} = 1$ ,  $\epsilon_d = -3$ ,  $U = 6$ , and  $\beta = 100$  is under the periodic Lorentzian pulses with the width  $\tau = 0.01$  and the frequency  $\hbar\Omega = 2$ .

be further analyzed with the electronic Wigner function

$$W(\omega, T) \equiv \frac{\Gamma_L \Gamma_R}{\Gamma_L + \Gamma_R} \text{Im} \int_{-\infty}^{\infty} dt_r e^{i\omega t_r} G^<(t, t'), \quad (5)$$

which provides information on both the spectral function and the nonequilibrium distribution of the photon-dressed electrons [29]. The lesser Green's function can also be efficiently calculated in the Floquet representation as  $\mathbf{G}^< = \mathbf{G}^r [\Sigma_0^< + \Sigma_U^<] \mathbf{G}^a$ . Although direct observation of the electronic Wigner function requires elaborate measurement setups [26, 29], it visualizes the real-time dynamics of the quasiparticles which are well represented in the energy domain.

The five panels in Fig. 2 show the electronic Wigner function of the photon-dressed electrons for various values of  $q$ . The characteristic structures which extend above the Fermi energy are fingerprints of levitons [26, 29]. The sharp Lorentzian pulses with  $\tau = 0.01$  and  $\hbar\Omega = 2$  create time-resolved excitations centered at  $t/T_p \in \mathbb{Z}$ . The singularities observed at  $\omega - \epsilon_F = n\hbar\Omega/2$  ( $n \in \mathbb{Z}$ ) result from the multiphoton-assisted transitions of the Fermi edge formed by the electrons in the left lead. At  $q = 0.6$  [Fig. 2(a)], the bare QD level with  $\epsilon_d = -3$  and  $\Gamma_{L,R} = 1$  is strongly perturbed by the Lorentzian pulses, resulting in the decoherence of the Kondo resonance [see also Fig. 1]. On the contrary, the weight of the DOS is gradually concentrated at the Fermi energy by increasing the pulse amplitude [Fig. 2(b)], and eventually becomes a stationary sharp peak at  $q = 1$  [Fig. 2(c)]. This counterintuitive emergence of the Kondo resonance under the strong driving field results from the special property of the Lorentzian pulses to minimize the disturbance of the Fermi sea at  $q \in \mathbb{Z}$ . Otherwise, a generic driving protocol produces particle-hole pairs, which inevitably inhibits the Kondo resonance. The coherent ripple patterns reported in Ref. 29 concurrently become clear at  $q = 1$  due to a quantum interference effect. When the amplitude becomes larger [Figs. 2(d) and (e)], the Kondo peak is smeared again because of both the dc offset of the pulses and the particle-hole excitations. The quantum ripples

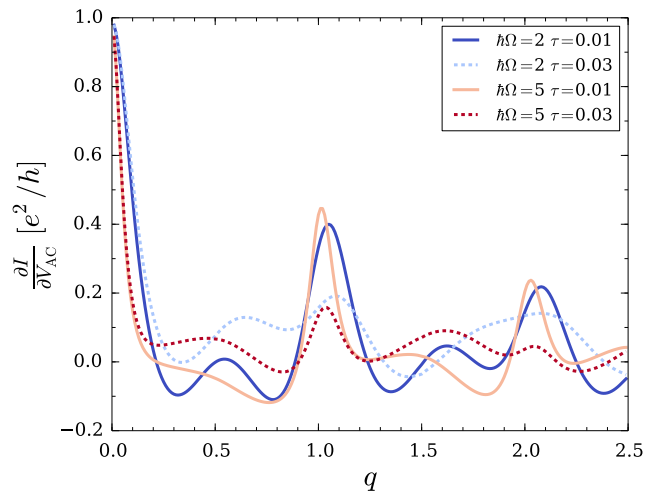


FIG. 3. (Color online) The dependence of the differential conductance  $\frac{\partial I}{\partial V_{AC}}$  on the amplitude  $q$  for various values of  $\tau$  and  $\Omega$ . The solid (dashed) lines correspond to  $\tau = 0.01$  ( $\tau = 0.03$ ) with  $\hbar\Omega = 2$  and  $\hbar\Omega = 5$ . The dot parameters are taken as  $\Gamma_{L,R} = 1$ ,  $\epsilon_d = -4$ ,  $U = 8$ , and  $\beta = 100$ .

also become obscure away from the optimal situation.

The coexistence of the leviton and the Kondo resonance results in enhancement of the dc current

$$I = \frac{e}{\hbar} \int d\omega \text{Tr} \left[ \left( \frac{-1}{\pi} \text{Im} \mathbf{G}^r(\omega) \right) (\mathbf{f}_L(\omega) - \mathbf{f}_R(\omega)) \right], \quad (6)$$

where the trace is taken over the Floquet indices. The dependence of the differential conductance  $\partial I / \partial V_{AC}$  on  $q$  is shown in Fig. 3 for various values of  $\tau$  and  $\Omega$ . At  $\tau = 0.01$ , the conductance is significantly enhanced around  $q \in \mathbb{Z}$  for both  $\hbar\Omega = 2$  and  $\hbar\Omega = 5$  cases, indicating the reformation of the many-body resonance in the dynamical regimes. The conductance at integer  $q$  is reduced for large values of  $\tau$  because the weight of the left conduction electrons forming the Kondo resonance decays as  $|u_{-1}|^2 \sim e^{-4\pi\tau}$ . The differential conductance shows rich

transport properties of levitons away from the optimal points as well. For instance, it can be negative between the resonant peaks due to the hole contribution generated by the non-quantized Lorentzian pulses. The peak found around  $q \simeq 0.5$  for  $\hbar\Omega = 2$  results from the photon-assisted transports: electrons can tunnel through the QD by absorbing photons. This picture is complementarily confirmed by the reduction of the corresponding peak for the case with  $\hbar\Omega = 5$ , where all the Floquet sidebands with positive quasienergies are located above the Fermi energy. The enhancement of the conductance at  $q \in \mathbb{Z}$  may provide a future experimental evidence of the leviton tunneling through the Kondo resonance. Recent experiments [25, 26] have succeeded in producing levitons with the driving frequency  $\Omega \simeq 38\text{GHz}$  at  $T_e \simeq 35\text{mK}$ , which is lower than the typical values of the Kondo temperature  $T_K \simeq 0.7\text{K} \sim 15\text{GHz}$  in a Kondo QD [5].

*Conclusion.*— In this Letter, we have demonstrated the coherent transport of levitons through the Kondo resonance realized in a QD system. The dynamical formation of the Kondo resonance and its coexistence with the levitons can be identified as the enhancement of the dc transport under quantized Lorentzian pulses. Since the leviton carries rich information on the many-body resonance state, we can probe the dynamical properties of the interacting system by measuring the quantum interference and the noise spectroscopy of the leviton. The present study also opens new possibilities for designing a quasiparticle excitation in interacting electron systems by engineering a time-dependent field.

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